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Integrable discretizations of the (2+1)-dimensional sinh-Gordon equation

Xing-Biao Hu[1] **and Guo-Fu Yu**[1,2]

[1] Institute of Computational Mathematics and Scientific Engineering Computing, Academy of Mathematics and System Sciences, Chinese Academy of Sciences, PO Box 2719, Beijing 100080, People's Republic of China

[2] Graduate School of the Chinese Academy of Sciences, Beijing, People's Republic of China

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Abstract
In this paper, we propose two semi-discrete equations and one fully discrete equation and study them by Hirota's bilinear method. These equations have continuum limits into a system which admits the (2+1)-dimensional generalization of the sinh-Gordon equation. As a result, two integrable semi-discrete versions and one fully discrete version for the sinh-Gordon equation are found. Bäcklund transformations, nonlinear superposition formulae, determinant solution and Lax pairs for these discrete versions are presented.

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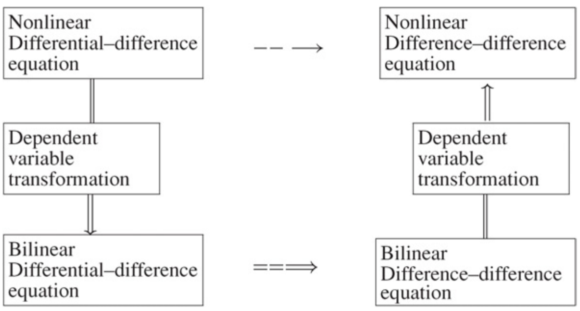
1. Introduction

Until now, much attention has been paid to the problem of integrable discretizations of integrable systems (see, e.g., [1–7] and references therein). It is highly nontrivial and of considerable interest to find integrable discretizations for integrable equations. Various approaches to the problem of integrable discretization are currently available. One of them is Hirota's approach [8]. Take an integrable differential–difference equation as an example. Hirota's method for discretizing soliton equations can be described via the following diagram [9]:



The sinh-Gordon equation $\omega_{xt} = \sinh\omega$ admits geometric interpretation as the differential equation which determines timelike surfaces of constant positive curvature in the same spaces. They appear in a wide range of physical applications including relativistic field theory, string dynamics, hydrodynamics, thermodynamics, solid-state physics and nonlinear optics [10]. The purpose of this paper is to consider integrable discretizations of bilinear version for the (2+1)-dimensional sinh-Gordon equation [11]:

$$\left[\frac{\mathrm{e}^{\omega}}{2}(\omega_{xt} - \sinh\omega)\right]_x + \frac{1}{2}\,\mathrm{e}^{\omega}\omega_{yt} = -\theta_x\,\mathrm{e}^{2\omega} - \mathrm{e}^{2\omega}\omega_x\theta, \tag{1.1}$$

$$(\theta_x\,\mathrm{e}^{\omega})_{xx} = -\frac{1}{2}\left[\omega_y + \omega_{xx} + \frac{1}{2}\omega_x^2\right]_{yt}, \tag{1.2}$$

which can be transformed into the following form under the dependent variable transformation $\upsilon \to \theta\,\mathrm{e}^{\omega}$,

$$(\omega_{xt} - \sinh\omega)_x + \omega_x(\omega_{xt} - \sinh\omega) + \omega_{yt} = -2\upsilon_x, \tag{1.3}$$

$$2(\upsilon_x - \upsilon\omega_x)_{xx} + \left(\omega_y + \omega_{xx} + \tfrac{1}{2}\omega_x^2\right)_{yt} = 0. \tag{1.4}$$

The paper is structured as follows. In section 2, an x-directional discrete version of the (2+1)-dimensional sinh-Gordon equation is found. A Bäcklund transformation, the corresponding nonlinear superposition formula, Lax pair and the Casorati determinant solution for this x-directional discrete system are shown. In section 3, a t-discrete version is found, its associated BT, superposition principle and Lax pair are given. Furthermore, a fully discrete version is worked out in section 4. It turns out that the resulting fully discrete version has a BT, nonlinear superposition formula and Lax pair. Conclusion and discussion are given in section 5. Finally, in appendix A we list some Hirota bilinear identities and in appendices B–D we will give detailed proofs of propositions 2, 4 and 6, respectively.

2. Integrable semi-discrete version in x-direction

Under the dependent variable transformation

$$\omega(x, y, t) = 2\ln\frac{F(x, y, t)}{G(x, y, t)}, \tag{2.1}$$

$$\theta(x, y, t) = \int_{-\infty}^{x} [\ln G(\xi, y, t)]_{yt}\left[\frac{G(\xi, y, t)}{F(\xi, y, t)}\right]^2 \mathrm{d}\xi, \tag{2.2}$$

equations (1.1) and (1.2) can be transformed into the following bilinear form [12]:

$$\left(D_y + D_x^2\right)F \bullet G = 0, \tag{2.3}$$

$$\left(D_x^2 D_t + D_y D_t + 2D_x\right)F \bullet G = 0. \tag{2.4}$$

When written in Hirota bilinear form, this system is seen to be part of the modified KP hierarchy with x and y of weight 1 and 2 as is usual, and t being of weight -1. In the following, for the sake of convenience, we introduce an additional discrete variable n and set

$$F = f_n, \qquad G = f_{n-1},$$

where we have denoted $F(x, y, t)$ to be F and $f(x, y, t, n)$ to be f_n for simplicity. Then equations (2.3) and (2.4) are reduced to

$$\left(D_y + D_x^2\right) \mathrm{e}^{\frac{D_n}{2}} f \bullet f = 0, \tag{2.5}$$

$$\left(D_x^2 D_t + D_y D_t + 2D_x\right) \mathrm{e}^{\frac{D_n}{2}} f \bullet f = 0. \tag{2.6}$$

In the following we would consider an integrable semi-discretization of (2.5) and (2.6). First we propose the following bilinear equations:

$$\left[D_y \,\mathrm{e}^{\frac{1}{2}D_n} + \frac{2}{\epsilon^2}\left(\mathrm{e}^{\epsilon D_x + \frac{1}{2}D_n} - \mathrm{e}^{\frac{1}{2}D_n}\right)\right] f_n \bullet f_n = 0, \tag{2.7}$$

$$\left[D_t D_y \,\mathrm{e}^{\frac{1}{2}D_n} + \frac{2}{\epsilon^2}\left(D_t \,\mathrm{e}^{\epsilon D_x + \frac{1}{2}D_n} - D_t \,\mathrm{e}^{\frac{1}{2}D_n}\right) + \frac{2}{\epsilon}\left(\mathrm{e}^{\frac{1}{2}D_n + \epsilon D_x} - \mathrm{e}^{\frac{1}{2}D_n}\right)\right] f_n \bullet f_n = 0. \tag{2.8}$$

It is not difficult to see that system (2.7), (2.8) has a singular limit into equations (2.5) and (2.6) through some variable transformation. In fact, by some calculations, it is shown that in the continuum limit as $\epsilon \to 0$, system (2.7), (2.8) is reduced to (2.5), (2.6) where $\partial_y + \frac{2}{\epsilon}\partial_x \longrightarrow \partial_y$. Therefore system (2.7), (2.8) serves as a x-directional discrete version for (2.5), (2.6). For simplicity we take $\epsilon = 1$ and rewrite variable x by m in the following discussion. In this case, system (2.7), (2.8) becomes

$$\left(D_y + 2\mathrm{e}^{D_m} - 2\right) \mathrm{e}^{\frac{D_n}{2}} f \bullet f = 0, \tag{2.9}$$

$$\left(D_t D_y + 2D_t \,\mathrm{e}^{D_m} - 2D_t + 2\mathrm{e}^{D_m} - 2\right) \mathrm{e}^{\frac{D_n}{2}} f \bullet f = 0. \tag{2.10}$$

By the dependent variable transformation

$$u_{m,n} = \ln \frac{f_{m+1}}{f}, \qquad v_{m,n} = \ln \frac{f_{m+1} f_{n-1}}{f_{m+1,n-1} f}, \tag{2.11}$$

the bilinear equations (2.5) and (2.6) can be transformed into the following nonlinear form:

$$v_y = 2\mathrm{e}^{v_{m-1} + u - u_{m-1}} - 2\mathrm{e}^{v + u_{m+1} - u}, \tag{2.12}$$

$$u_{ty} + (2u_{m+1,t} + 1)\,\mathrm{e}^{u_{m+1} - u + v} - (2u_t + 1)\,\mathrm{e}^{u - u_{m-1} + v_{m-1}} = 0. \tag{2.13}$$

It should be mentioned that the dependence on n is not genuine in equations (2.12) and (2.13) (there is no shift in variable n in the equations). In equations (2.12) and (2.13) and in the following we will always use a simplified notation for the functions u, v etc. We write explicitly a discrete independent variable only when it is shifted from its position.

Since we are looking for integrable discretization of the (2+1)-dimensional generalization of the sinh-Gordon equation, we need to show integrability of equations (2.9) and (2.10). The justification to this is via Bäcklund transformation and Lax pair. We will show that equations (2.9) and (2.10) are integrable in the sense of having a Bäcklund transformation and Lax pair. Concerning the bilinear equations (2.9) and (2.10), we have the following result:

Proposition 1. *A Bäcklund transformation for (2.9) and (2.10) is*

$$\left(\lambda \,\mathrm{e}^{\frac{D_n}{2} - \frac{D_m}{2}} + \mu \,\mathrm{e}^{-\frac{D_n}{2} - \frac{D_m}{2}} - \mathrm{e}^{\frac{D_n}{2} + \frac{D_m}{2}}\right) f \bullet g = 0, \tag{2.14}$$

$$\left(D_y - 2\lambda \,\mathrm{e}^{-D_m} + \theta\right) f \bullet g = 0, \tag{2.15}$$

$$\left(D_t\,\mathrm{e}^{\frac{D_m}{2}+\frac{D_n}{2}} - \lambda D_t\,\mathrm{e}^{-\frac{D_m}{2}+\frac{D_n}{2}} + \mu D_t\,\mathrm{e}^{-\frac{D_m}{2}-\frac{D_n}{2}} + \mathrm{e}^{\frac{D_m}{2}+\frac{D_n}{2}} + \gamma\,\mathrm{e}^{-\frac{D_m}{2}-\frac{D_n}{2}}\right) f \bullet g = 0. \tag{2.16}$$

where λ, μ, γ *and* θ *are arbitrary constants.*

Proof. Let $f(m,n)$ be a solution of equations (2.9) and (2.10). If we can show that equations (2.14)–(2.16) guarantee that the following two relations:

$$P_1 \equiv \left[(D_y + 2\mathrm{e}^{D_m} - 2)\,\mathrm{e}^{\frac{D_n}{2}}\right] g(m,n) \bullet g(m,n) = 0, \tag{2.17}$$

$$P_2 \equiv \left[(D_t D_y + 2D_t\,\mathrm{e}^{D_m} - 2D_t + 2\mathrm{e}^{D_m} - 2)\,\mathrm{e}^{\frac{D_n}{2}}\right] g(m,n) \bullet g(m,n) = 0, \tag{2.18}$$

hold, then equations (2.14)–(2.16) form a *Bäcklund* transformation. □

By use of (2.14)–(2.16), (A.1), (A.2), (A.3) and (A.5), we have

$$\begin{aligned}
-\left(\mathrm{e}^{\frac{1}{2}D_n} f \bullet f\right) P_1 &= 2\sinh\left(\frac{1}{2}D_n\right)(D_y f \bullet g) \bullet (fg) \\
&\quad + 4\sinh\left(\frac{1}{2}D_m\right)\left(\mathrm{e}^{\frac{D_n}{2}+\frac{D_m}{2}} f \bullet g\right) \bullet \left(\mathrm{e}^{-\frac{D_n}{2}-\frac{D_m}{2}} f \bullet g\right) \\
&= 2\sinh\left(\frac{D_n}{2}\right)(D_y f \bullet g) \bullet (fg) \\
&\quad + 4\lambda\sinh\left(\frac{D_m}{2}\right)\left(\mathrm{e}^{\frac{D_n}{2}-\frac{D_m}{2}} f \bullet g\right) \bullet \left(\mathrm{e}^{-\frac{D_n}{2}-\frac{D_m}{2}} f \bullet g\right) \\
&= 2\sinh\left(\frac{D_n}{2}\right)[(D_y - 2\lambda\,\mathrm{e}^{-D_m})(f \bullet g)] \bullet (fg) = 0.
\end{aligned}$$

Thus we have proved that (2.17) holds. Similarly

$$\begin{aligned}
-\left(\mathrm{e}^{\frac{D_n}{2}} f \bullet f\right) P_2 &= \left[\left(D_t D_y\,\mathrm{e}^{\frac{1}{2}D_n} + 2D_t\,\mathrm{e}^{D_m+\frac{1}{2}D_n} - 2D_t\,\mathrm{e}^{\frac{1}{2}D_n} + 2\mathrm{e}^{D_m+\frac{1}{2}D_n} - 2\mathrm{e}^{\frac{1}{2}D_n}\right) f \bullet f\right] \\
&\quad \times \left[\mathrm{e}^{\frac{1}{2}D_n} g \bullet g\right] - \left[\left(D_t D_y\,\mathrm{e}^{\frac{1}{2}D_n} + 2D_t\,\mathrm{e}^{D_m+\frac{1}{2}D_n} - 2D_t\,\mathrm{e}^{\frac{1}{2}D_n}\right.\right. \\
&\quad \left.\left. + 2\mathrm{e}^{D_m+\frac{1}{2}D_n} - 2\mathrm{e}^{\frac{1}{2}D_n}\right) g \bullet g\right]\left[\mathrm{e}^{\frac{1}{2}D_n} f \bullet f\right] \\
&\quad + \left[\left(D_y\,\mathrm{e}^{\frac{1}{2}D_n} + 2\mathrm{e}^{D_m+\frac{1}{2}D_n} - 2\mathrm{e}^{\frac{1}{2}D_n}\right) f \bullet f\right]\left[D_t\,\mathrm{e}^{\frac{1}{2}D_n} g \bullet g\right] \\
&\quad - \left[\left(D_y\,\mathrm{e}^{\frac{1}{2}D_n} + 2\mathrm{e}^{D_m+\frac{1}{2}D_n} - 2\mathrm{e}^{\frac{1}{2}D_n}\right) g \bullet g\right]\left[D_t\,\mathrm{e}^{\frac{1}{2}D_n} f \bullet f\right] \\
&= 2D_t\cosh\left(\tfrac{1}{2}D_n\right)(D_y f \bullet g) \bullet (fg) + 4\sinh\left(\tfrac{1}{2}D_m\right) \\
&\quad \times \left\{\left(D_t\,\mathrm{e}^{\frac{1}{2}D_n+\frac{1}{2}D_m} f \bullet g\right) \bullet \left(\mathrm{e}^{-\frac{1}{2}D_m-\frac{1}{2}D_n} f \bullet g\right)\right. \\
&\quad \left. - \left(\mathrm{e}^{\frac{1}{2}D_m+\frac{1}{2}D_n} f \bullet g\right) \bullet \left(D_t\,\mathrm{e}^{-\frac{1}{2}D_m-\frac{1}{2}D_n} f \bullet g\right)\right\} \\
&\quad + 4\sinh\left(\tfrac{1}{2}D_m\right)\left(\mathrm{e}^{\frac{1}{2}D_m+\frac{1}{2}D_n}\right) \bullet \left(\mathrm{e}^{-\frac{1}{2}D_m-\frac{1}{2}D_n} f \bullet g\right) \\
&= 4\lambda D_t\cosh\left(\tfrac{1}{2}D_n\right)\left(\mathrm{e}^{D_m} f \bullet g\right) \bullet (fg) + 4\sinh\left(\tfrac{1}{2}D_n\right) \\
&\quad \times \left\{\left(D_t\,\mathrm{e}^{\frac{1}{2}D_m+\frac{1}{2}D_n} f \bullet g\right) \bullet \left(\mathrm{e}^{-\frac{1}{2}D_m-\frac{1}{2}D_n} f \bullet g\right)\right. \\
&\quad \left. - \left(\mathrm{e}^{\frac{1}{2}D_m+\frac{1}{2}D_n} f \bullet g\right) \bullet \left(D_t\,\mathrm{e}^{-\frac{1}{2}D_m-\frac{1}{2}D_n}\right) f \bullet g\right\} \\
&\quad + 4\sinh\left(\tfrac{1}{2}D_m\right)\left(\mathrm{e}^{\frac{1}{2}D_m+\frac{1}{2}D_n}\right) \bullet \left(\mathrm{e}^{-\frac{1}{2}D_m-\frac{1}{2}D_n} f \bullet g\right) \\
&= 4\sinh\left(\tfrac{1}{2}D_m\right)\left[\left(-\lambda D_t\,\mathrm{e}^{-\frac{1}{2}D_m+\frac{1}{2}D_n} + D_t\,\mathrm{e}^{\frac{1}{2}D_m+\frac{1}{2}D_n}\right.\right. \\
&\quad \left.\left. + \mu D_t\,\mathrm{e}^{-\frac{1}{2}D_n-\frac{1}{2}D_m} + \mathrm{e}^{\frac{1}{2}D_m+\frac{1}{2}D_n} f \bullet g\right)\right] \bullet \left(\mathrm{e}^{-\frac{1}{2}D_m-\frac{1}{2}D_n} f \bullet g\right) \\
&= 0.
\end{aligned}$$

Thus we have completed the proof of proposition 1.

Concerning the BT (2.14)–(2.16), we have the following result:

Proposition 2. *Let f_0 be a solution of the semi-discrete sinh-Gordon equations (2.9) and (2.10). Suppose that f_i $(i=1,2)$ is another solution of (2.9) and (2.10) which is related to f_0 under BT (2.14)–(2.16) with parameters $(\lambda_i,\mu_i,\gamma_i,\theta_i)$, i.e., $f_0 \stackrel{(\lambda_i,\mu_i,\gamma_i,\theta_i)}{\longrightarrow} f_i$ $(i=1,2)$, $\lambda_1\lambda_2\neq 0$, $f_i\neq 0$ $(i=0,1,2)$. Then f_{12} defined by*

$$\kappa\,\mathrm{e}^{-\frac{1}{2}D_m} f_0\bullet f_{12} = \left[\lambda_1\,\mathrm{e}^{\frac{1}{2}D_m} - \lambda_2\,\mathrm{e}^{-\frac{1}{2}D_m}\right] f_1\bullet f_2 \tag{2.19}$$

with κ being a non-zero constant, is a new solution which is related to f_1 and f_2 under the BT (2.14)–(2.16) with parameters $(\lambda_1,\mu_1,\gamma_1,\theta_1)$, $(\lambda_2,\mu_2,\gamma_2,\theta_2)$, respectively.

We will give a detailed proof of proposition 2 in appendix B.

Starting from the bilinear BT (2.14)–(2.16), we can derive a Lax pair for the system (2.12), (2.13). Firstly, set

$$\Psi=\frac{f}{g},\qquad \Phi=\frac{f_{n-1}}{g_{n-1}},\qquad u=\ln\frac{g_{m+1}}{g},\qquad v=\ln\frac{g_{m+1}g_{n-1}}{g_{m+1,n-1}g}$$

in (2.14)–(2.16). Then from the bilinear BT (2.14)–(2.16) and after some calculations, we can obtain the following Lax pair for (2.12) and (2.13):

$$\lambda\,\mathrm{e}^{-v}\Psi-\Psi_{m+1}+\mu\Phi=0, \tag{2.20}$$

$$\Psi_y=2\lambda\,\mathrm{e}^{u-u_{m-1}}\Psi_{m-1}-\theta\Psi, \tag{2.21}$$

$$\Phi_y=2\lambda\,\mathrm{e}^{u-u_{m-1}-v+v_{m-1}}\Phi_{m-1}-\theta\Phi, \tag{2.22}$$

$$\Psi_{m+1,t}-\lambda\,\mathrm{e}^{-v}\Psi_t+\frac{1}{2}\left(1+\frac{\gamma}{\mu}\right)\Psi_{m+1}+\lambda u_t\,\mathrm{e}^{-v}\Psi-\frac{\lambda\gamma}{2\mu}\,\mathrm{e}^{-v}\Psi=0. \tag{2.23}$$

We can show that the compatibility conditions of (2.20)–(2.23) yield the semi-discrete system (2.12), (2.13), so they constitute a Lax pair for the system (2.9), (2.10) or (2.12), (2.13).

Since the (2+1)-dimensional sinh-Gordon equation has a Wronskian determinant solution, we would expect the discretized system should keep the determinant structure of solutions. In the following we would give the Casorati determinant solution for bilinear equations (2.9) and (2.10). Following Nimmo and Freeman's notation [13, 14], let us denote

$$f_{m,n}=|0_n,1_n,\ldots,N-1_n|=\begin{vmatrix}\phi_1(n;m) & \phi_1(n;m+1) & \cdots & \phi_1(n;m+N-1)\\ \phi_2(n;m) & \phi_2(n;m+1) & \cdots & \phi_2(n;m+N-1)\\ \vdots & \vdots & & \vdots\\ \phi_N(n;m) & \phi_N(n;m+1) & \cdots & \phi_N(n;m+N-1)\end{vmatrix}, \tag{2.24}$$

where $\phi_i(n;k)$, $k=m,m+1,\ldots,m+N-1$ satisfy the dispersion relations

$$\frac{\partial}{\partial t}\left[\phi_i(n;k+1)+\phi_i(n;k)\right]=\frac{1}{2}\phi_i(n;k), \tag{2.25}$$

$$\frac{\partial}{\partial y}\phi_i(n;k)=2\phi_i(n;k+1), \tag{2.26}$$

$$\phi_i(n+1;k)=\phi_i(n;k)+\phi_i(n;k-1). \tag{2.27}$$

A particular solution of (2.25)–(2.27) is obtained by choosing 'exponential type' functions as

$$\phi_i(n;m) = c_i p_i^m \left(1+p_i^{-1}\right)^n e^{2p_i y + \frac{t}{2+2p_i}} + d_i q_i^m \left(1+q_i^{-1}\right)^n e^{2q_i y + \frac{t}{2+2q_i}}, \tag{2.28}$$

where $c_i, d_i, p_i \neq 0, q_i \neq 0$ are arbitrary parameters. Equations (2.9) and (2.10) can be written as

$$f_{n,y} f_{n-1} - f_n f_{n-1,y} + 2 f_{m+1} f_{m-1,n-1} - 2 f_n f_{n-1} = 0, \tag{2.29}$$

$$\begin{aligned} &f_{ty} f_{n-1} - f_t f_{n-1,y} - f_y f_{n-1,t} + f f_{n-1,ty} + 2 f_{m+1,t} f_{m-1,n-1} - 2 f_{m+1} f_{m-1,n-1,t} \\ &\quad - 2 f_t f_{n-1} + 2 f f_{n-1,t} + 2 f_{m+1} f_{m-1,n-1} - 2 f_n f_{n-1} = 0. \end{aligned} \tag{2.30}$$

By using the dispersion relation (2.25)–(2.27), we can obtain expressions for the terms involved in (2.29) and (2.30) as

$$\begin{aligned} &f = |0, 1, \ldots, N-1| = |1_{n+1}, 2_{n+1}, \ldots, N-1_{n+1}, N-1| = |0, 1_{n+1}, \ldots, N-1_{n+1}|, \\ &f_{n-1} = |1, 2, \ldots, N-1, N-1_{n-1}| = |0_{n-1}, 1, \ldots, N-1|, \\ &\qquad f_{m+1} = |1, 2, \ldots, N-1, N|, \\ &f_{n-1,m-1} = |0, 1, \ldots, N-2, N-1| - |0, 1, \ldots, N-2, N-1_{n-1}|, \\ &f_y = 2|0, 1, \ldots, N-2, N|, \qquad f_{n-1,t} = |0_{n-1,t}, 1, 2, \ldots, N-1|, \\ &f_{n-1,y} = 2|1, 2, \ldots, N-2, N, N-1_{n-1}| \\ &\qquad + 2|1, 2, \ldots, N| - 2|1, 2, \ldots, N-1, N-1_{n-1}|, \\ &f_t = \tfrac{1}{2}|0, 1_{n+1}, \ldots, N-1_{n+1}| - \tfrac{1}{2}|0_{n-1}, 1_{n+1}, 2_{n+1}, \ldots, N-1_{n+1}|, \\ &f_{ty} = |0_{n-1}, 1_{n+1}, \ldots, N-1_{n+1}| + |0, 1_{n+1}, \ldots, N-2_{n+1}, N_{n+1}| \\ &\qquad - |0_{n-1}, 1_{n+1}, \ldots, N-2_{n+1}, N_{n+1}|, \\ &f_{m+1} = |1, 2, \ldots, N-1, N|, \qquad f_{m+1,t} = \tfrac{1}{2}|0_{n-1}, 2_{n+1}, \ldots, N_{n+1}|. \end{aligned} \tag{2.31}$$

Using these expressions, we see that (2.9) reduces to the Plücker relation [15]:

$$\begin{aligned} &2|0, 1, \ldots, N-2, N||1, 2, \ldots, N-2, N-1, N-1_{n-1}| \\ &\quad - 2|0, 1, \ldots, N-2, N-1||1, \ldots, N-2, N, N-1_{n-1}| \\ &\quad - 2|1, \ldots, N-2, N-1, N||0, 1, \ldots, N-2, N-1_{n-1}| = 0 \end{aligned} \tag{2.32}$$

and (2.10) also reduces to the sum of the following three Plücker relations:

$$\begin{aligned} &|0_{n-1}, 1, \ldots, N-2, N-1||0, 1, \ldots, N-2, N| - |0_{n-1}, 1, \ldots, N-2, N||0, 1, \ldots, N-1| \\ &\quad - |1, 2, \ldots, N||0_{n-1}, 0, 1, \ldots, N-2| \\ &\quad + |0_{n-1}, 2_{n+1}, \ldots, N-1_{n+1}, N-2||0_{n-1}, 1_{n+1}, \ldots, N-2_{n+1}, N_{n+1}| \\ &\quad - |0_{n-1}, 2_{n+1}, \ldots, N-1_{n+1}, N_{n+1}||0_{n-1}, 1_{n+1}, \ldots, N-2_{n+1}, N-2| \\ &\quad + |0_{n-1}, 2_{n+1}, \ldots, N-2_{n+1} N-2, N_{n+1}||0_{n-1}, 1_{n+1}, \ldots, N-2_{n+1}, N-1_{n+1}| \\ &\quad + |0, 1, \ldots, N-1||0_{n-1,t}, 1, \ldots, N-2, N| \\ &\quad + |1, \ldots, N||0_{n-1,t}, 0, 1, \ldots, N-2| \\ &\quad - |0_{n-1,t}, 1, 2, \ldots, N-1||0, 1, \ldots, N-2, N| = 0. \end{aligned} \tag{2.33}$$

Therefore, the semi-discrete version (2.9) and (2.10) has the Casorati determinant solution (2.24) with dispersion relations (2.25)–(2.27).

3. Integrable semi-discrete version in *t*-direction

Similar to section 2, we first propose the following bilinear equations:

$$\left[D_y\, \mathrm{e}^{\frac{1}{2}D_n} + 2\left(\mathrm{e}^{D_m+\frac{1}{2}D_n} - \mathrm{e}^{\frac{1}{2}D_n}\right)\right] f \bullet f = 0, \tag{3.1}$$

$$\left[\frac{1}{\epsilon} D_y\, \mathrm{e}^{\frac{1}{2}D_n+\epsilon D_t} + \frac{2}{\epsilon}\, \mathrm{e}^{D_m+\frac{1}{2}D_n+\epsilon D_t} - \frac{2}{\epsilon}\, \mathrm{e}^{\frac{1}{2}D_n+\epsilon D_t} + 2\mathrm{e}^{\frac{1}{2}D_n+D_m+\epsilon D_t} - 2\mathrm{e}^{\frac{1}{2}D_n+\epsilon D_t}\right] f \bullet f = 0. \tag{3.2}$$

By some calculations, it is shown that in the continuum limit as $\epsilon \to 0$, the system (3.1), (3.2) is reduced to (2.9), (2.10). Therefore the system (3.1), (3.2) serves as a t-direction discrete version for (2.9), (2.10) or (2.5), (2.6). For simplicity we take $\epsilon = 1$ and rewrite variable t by k in the following discussion. In this case, the system (3.1), (3.2) becomes

$$\left[D_y\, \mathrm{e}^{\frac{1}{2}D_n} + 2\left(\mathrm{e}^{D_m+\frac{1}{2}D_n} - \mathrm{e}^{\frac{1}{2}D_n}\right)\right] f \bullet f = 0, \tag{3.3}$$

$$\left[D_y\, \mathrm{e}^{\frac{1}{2}D_n+D_k} + 4\mathrm{e}^{D_m+\frac{1}{2}D_n+D_k} - 4\mathrm{e}^{\frac{1}{2}D_n+D_k}\right] f \bullet f = 0. \tag{3.4}$$

By the dependent variable transformation

$$u = \ln \frac{f_{m+1}}{f}, \qquad v = \ln \frac{f_{m+1} f_{n+1}}{f_{m+1,n+1} f}, \tag{3.5}$$

the bilinear equations (3.3) and (3.4) can be transformed into the following nonlinear form:

$$v_y - 2\mathrm{e}^{u_{m+1}-u-v} + 2\mathrm{e}^{u-u_{m-1}-v_{m-1}} = 0, \tag{3.6}$$

$$u_{k+1,y} - u_{k-1,y} + \left(4\mathrm{e}^{u_{m+1,k+1}-u_{k-1}-v_{k-1}} - 2\mathrm{e}^{u_{m+1,k-1}-u_{k-1}-v_{k-1}}\right) - \left(4\mathrm{e}^{u_{k+1}-u_{m-1,k-1}-v_{m-1,k-1}} - 2\mathrm{e}^{u_{k-1}-u_{m-1,k-1}-v_{m-1,k-1}}\right) = 0. \tag{3.7}$$

It is also remarked that the dependence on n is not genuine in equations (3.6) and (3.7) (there is no shift in variable n in the equations).

Proposition 3. *A Bäcklund transformation for (3.3) and (3.4) is*

$$\left(\lambda\, \mathrm{e}^{\frac{D_n}{2}-\frac{D_m}{2}} + \mu\, \mathrm{e}^{-\frac{D_n}{2}-\frac{D_m}{2}} - \mathrm{e}^{\frac{D_n}{2}+\frac{D_m}{2}}\right) f \bullet g = 0, \tag{3.8}$$

$$\left(D_y - 2\lambda\, \mathrm{e}^{-D_m} + \theta\right) f \bullet g = 0, \tag{3.9}$$

$$\left(-\lambda\, \mathrm{e}^{-\frac{D_m}{2}+\frac{D_n}{2}+D_k} + 2\mathrm{e}^{\frac{D_m}{2}+\frac{D_n}{2}+D_k} + \gamma\, \mathrm{e}^{-\frac{D_m}{2}-\frac{D_n}{2}-D_k}\right) f \bullet g = 0 \tag{3.10}$$

where λ, μ, γ and θ are arbitrary constants.

Proof. Let $f(m,n)$ be a solution of equations (3.3) and (3.4). If we can show that equations (3.8)–(3.10) guarantee that the following two relations:

$$P_1 \equiv \left[\left(D_y + 2\mathrm{e}^{D_m} - 2\right) \mathrm{e}^{\frac{D_n}{2}}\right] g(m,n,k) \bullet g(m,n,k) = 0, \tag{3.11}$$

$$P_2 \equiv \left[D_y\, \mathrm{e}^{\frac{1}{2}D_n+D_k} + 4\mathrm{e}^{D_m+\frac{1}{2}D_n+D_k} - 4\mathrm{e}^{\frac{1}{2}D_n+D_k}\right] g(m,n,k) \bullet g(m,n,k) = 0, \tag{3.12}$$

hold, then equations (3.8)–(3.10) form a *Bäcklund* transformation. □

By use of (3.8)–(3.10), (A.1), (A.4) and (A.6), we have

$$\begin{aligned}
-\left(e^{\frac{1}{2}D_n+D_k} f \bullet f\right) P_2 &= 2\sinh\left(\frac{1}{2}D_n + D_k\right)(D_y f \bullet g) \bullet (fg) \\
&\quad + 8\sinh\left(\frac{1}{2}D_m\right)\left(e^{\frac{D_n}{2}+\frac{D_m}{2}+D_k} f \bullet g\right) \bullet \left(e^{-\frac{D_n}{2}-\frac{D_m}{2}-D_k} f \bullet g\right) \\
&= 4\sinh\left(\frac{D_m}{2}\right)\left[\left(-\lambda\, e^{-\frac{1}{2}D_m+\frac{1}{2}D_n+D_k} + 2e^{\frac{1}{2}D_m+\frac{1}{2}D_n+D_k}\right) f \bullet g\right] \\
&\quad \bullet \left(e^{-\frac{1}{2}D_m-\frac{1}{2}D_n-D_k} f \bullet g\right) = 0.
\end{aligned}$$

Thus we have proved that (3.12) holds. Similarly we can show that (3.11) holds. Thus we have completed the proof of proposition 3.

Concerning the BT (3.8)–(3.10), we have the following result:

Proposition 4. *Let f_0 be a solution of the semi-discrete sinh-Gordon equations (3.3) and (3.4). Suppose that f_i $(i = 1, 2)$ is another solution of (3.3)–(3.4) which is related to f_0 under BT (3.8)–(3.10) with parameters $(\lambda_i, \mu_i, \gamma_i, \theta_i)$, i.e., $f_0 \overset{(\lambda_i,\mu_i,\gamma_i,\theta_i)}{\longrightarrow} f_i$ $(i = 1, 2), \lambda_1\lambda_2 \neq 0$, $f_i \neq 0$ $(i = 0, 1, 2)$. Then f_{12} defined by*

$$k\, e^{-\frac{1}{2}D_m} f_0 \bullet f_{12} = \left[\lambda_2\, e^{-\frac{1}{2}D_m} - \lambda_1\, e^{\frac{1}{2}D_m}\right] f_1 \bullet f_2 \tag{3.13}$$

with k being a non-zero constant, is a new solution which is related to f_1 and f_2 under the BT (3.8)–(3.10) with parameters $(\lambda_1, \mu_1, \gamma_1, \theta_1)$, $(\lambda_2, \mu_2, \gamma_2, \theta_2)$, respectively.

We will give a detailed proof of proposition 4 in appendix C.

Starting from the bilinear BT (3.8)–(3.10), we can derive a Lax pair for the system (3.6), (3.7). Firstly, set

$$\psi = \frac{f}{g}, \qquad u = \ln\frac{g_{m+1}}{g}, \qquad v = \ln\frac{g_{m+1,n-1}g}{g_{m+1}g_{n-1}}$$

in (3.8)–(3.10). Then from the bilinear BT (3.8)–(3.10) and after some calculations, we can obtain the following Lax pair for (3.6) and (3.7):

$$\lambda\mu\, e^{u_{k-1}-u_{k+1}+v_{k-1}}\psi_{k+1} - 2\mu\psi_{m+1,k+1} - \gamma\psi_{m+1,k-1} + \lambda\gamma\, e^{v_{k-1}}\psi_{k-1} = 0, \tag{3.14}$$

$$\psi_y - 2\lambda\psi_{m-1}\, e^{u-u_{m-1}} + \theta\psi = 0. \tag{3.15}$$

We have checked that the compatibility condition of (3.14) and (3.15) yields the semi-discrete system (3.6), (3.7).

4. Integrable fully discrete version

In this section we would propose fully discrete version of the (2+1)-dimensional sinh-Gordon equation. First of all we give the bilinear equations

$$\left[\frac{1}{2\epsilon}\, e^{\frac{1}{2}D_n+\epsilon D_y} - \frac{1}{2\epsilon}\, e^{\frac{1}{2}D_n-\epsilon D_y} + 2e^{D_m+\frac{1}{2}D_n+\epsilon D_y} - 2e^{\frac{1}{2}D_n-\epsilon D_y}\right] f \bullet f = 0, \tag{4.1}$$

$$\left[\frac{1}{2\epsilon}\, e^{\frac{1}{2}D_n+D_k+\epsilon D_y} - \frac{1}{2\epsilon}\, e^{\frac{1}{2}D_n+D_k-\epsilon D_y} + 4e^{D_m+\frac{1}{2}D_n+D_k+\epsilon D_y} - 4e^{\frac{1}{2}D_n+D_k-\epsilon D_y}\right] f \bullet f = 0. \tag{4.2}$$

In the continuum limit as $\epsilon \to 0$, the system (4.1), (4.2) is reduced to semi-discrete equations (3.3) and (3.4). Therefore the system (4.1), (4.2) serves as a full-discrete version for (2.5), (2.6). For simplicity we take $\epsilon = 1$ and rewrite variable y by l in the following discussion. In this case the system (4.1), (4.2) becomes

$$\left[e^{\frac{1}{2}D_n+D_l} - 5e^{\frac{1}{2}D_n-D_l} + 4e^{D_m+\frac{1}{2}D_n+D_l}\right] f \bullet f = 0, \tag{4.3}$$

$$\left[e^{\frac{1}{2}D_n+D_k+D_l} - 9e^{\frac{1}{2}D_n+D_k-D_l} + 8e^{D_m+\frac{1}{2}D_n+D_k+D_l}\right] f \bullet f = 0. \tag{4.4}$$

If we let $F = f$, $G = f_{n-1}$, equations (4.3) and (4.4) are equivalent to

$$(e^{D_l} - 5e^{-D_l} + 4e^{D_m+D_l}) F \bullet G = 0, \tag{4.5}$$

$$(e^{D_k+D_l} - 9e^{D_k-D_l} + 8e^{D_m+D_k+D_l}) F \bullet G = 0. \tag{4.6}$$

When written in the above Hirota bilinear form, this system is seen to be part of the modified discrete KP hierarchy. Through the dependent variable transformation

$$u = \ln \frac{F_{m+1}}{F}, \qquad v = \ln \frac{G_{m+1} F}{F_{m+1} G} \tag{4.7}$$

the bilinear equations (4.5) and (4.6) can be transformed into the following nonlinear form:

$$4e^{u_{m+1,l+1}-u_{l-1}} - 4e^{u_{l+1}-u_{m-1,l-1}+v_{l+1}-v_{m-1,l-1}} + e^{v_{l-1}} - e^{v_{l+1}} = 0, \tag{4.8}$$

$$e^{v_{k-1,l-1}+u_{k-1,l-1}-u_{k+1,l-1}} - e^{v_{k-1,l+1}+u_{k-1,l+1}-u_{k+1,l+1}} - 8e^{u_{m+1,k-1,l+1}-u_{k-1,l-1}} + 8e^{u_{m+1,k+1,l+1}-u_{k+1,l-1}} - 2e^{v_{k-1,l-1}} + 2e^{v_{k-1,l+1}} = 0. \tag{4.9}$$

Proposition 5. *A Bäcklund transformation for (4.3) and (4.4) is*

$$\left(\lambda\, e^{\frac{D_n}{2}-\frac{D_m}{2}} + \mu\, e^{-\frac{D_n}{2}-\frac{D_m}{2}} - e^{\frac{D_n}{2}+\frac{D_m}{2}}\right) f \bullet g = 0, \tag{4.10}$$

$$\left(e^{-D_l} + 4\lambda\, e^{-D_m-D_l} + \theta\, e^{D_l}\right) f \bullet g = 0, \tag{4.11}$$

$$\left(-\lambda\, e^{-\frac{D_m}{2}+\frac{D_n}{2}+D_k} + 2e^{\frac{D_m}{2}+\frac{D_n}{2}+D_k} + \alpha\, e^{\frac{-D_m}{2}-\frac{D_n}{2}-D_k}\right) f \bullet g = 0 \tag{4.12}$$

where λ, μ, α and θ are arbitrary constants.

Proof. Let $f(m,n)$ be a solution of equations (4.3) and (4.4). If we can show that equations (4.10)–(4.12) guarantee that the following two relations,

$$P_1 \equiv \left[e^{\frac{1}{2}D_n+D_l} - 5e^{\frac{1}{2}D_n-D_l} + 4e^{D_m+\frac{1}{2}D_n+D_l}\right] g(m,n,k,l) \bullet g(m,n,k,l) = 0, \tag{4.13}$$

$$P_2 \equiv \left[e^{\frac{1}{2}D_n+D_k+D_l} - 9e^{\frac{1}{2}D_n+D_k-D_l} + 8e^{D_m+\frac{1}{2}D_n+D_k+D_l}\right] g(m,n,k,l) \bullet g(m,n,k,l) = 0, \tag{4.14}$$

hold, then equations (4.10)–(4.12) form a *Bäcklund* transformation. By use of (4.10)–(4.12), (A.6), we have

$$\begin{aligned}
-\left(e^{\frac{D_n}{2}-D_l} f \bullet f\right) P_1 &= 2\sinh\left(\frac{D_n}{2}\right) (e^{D_l} f \bullet g) \bullet (e^{-D_l} f \bullet g) \\
&\quad + 8\lambda \sinh\left(\frac{D_m}{2} + D_l\right) (e^{\frac{D_n}{2}-\frac{D_m}{2}} f \bullet g) \bullet (e^{-\frac{D_n}{2}-\frac{D_m}{2}} f \bullet g) \\
&= 2\sinh\left(\frac{D_n}{2}\right) (e^{D_l} f \bullet g) \bullet \left[\left(e^{-D_l} + 4\lambda\, e^{-D_m-D_l} f \bullet g\right)\right] = 0.
\end{aligned}$$

Thus we have proved that (4.13) holds. Similarly

$$
\begin{aligned}
-\left(\mathrm{e}^{\frac{1}{2}D_n+D_k-D_l} f \bullet f\right) P_2 &= 2\sinh\left(\frac{D_n}{2}+D_k\right)(\mathrm{e}^{D_l} f \bullet g)\bullet(\mathrm{e}^{-D_l} f \bullet g)\\
&\quad + 16\sinh\left(\frac{D_m}{2}+D_l\right)\left(\mathrm{e}^{\frac{D_n}{2}+\frac{D_m}{2}+D_k} f \bullet g\right)\bullet\left(\mathrm{e}^{-\frac{D_n}{2}-\frac{D_m}{2}-D_k} f \bullet g\right)\\
&= -8\lambda\sinh\left(\frac{D_n}{2}+D_k\right)(\mathrm{e}^{D_l} f \bullet g)\bullet(\mathrm{e}^{-D_m-D_l} f \bullet g)\\
&\quad + 16\sinh\left(\frac{D_m}{2}+D_l\right)\left(\mathrm{e}^{\frac{D_n}{2}+\frac{D_m}{2}+D_k} f \bullet g\right)\bullet\left(\mathrm{e}^{-\frac{D_n}{2}-\frac{D_m}{2}-D_k} f \bullet g\right)\\
&= 8\sinh\left(\frac{D_m}{2}+D_l\right)\left[\left(-\lambda\,\mathrm{e}^{-\frac{1}{2}D_m+\frac{1}{2}D_n+D_k}+2\mathrm{e}^{\frac{1}{2}D_m+\frac{1}{2}D_n+D_k}\right) f \bullet g\right]\\
&\quad \bullet\left(\mathrm{e}^{-\frac{1}{2}D_m-\frac{1}{2}D_n-D_k} f \bullet g\right) = 0.
\end{aligned}
$$

Thus we have proved that (4.14) holds. Thus we have completed the proof of proposition 5. □

Concerning the BT (4.10)–(4.12), we have the following result:

Proposition 6. *Let f_0 be a solution of the fully discrete sinh-Gordon equations (4.3) and (4.4). Suppose that f_i $(i = 1, 2)$ is another solution of (4.3)–(4.4) which is related to f_0 under BT (4.10)–(4.12) with parameters $(\lambda_i, \mu_i, \gamma_i, \theta_i)$, i.e., $f_0 \xrightarrow{(\lambda_i,\mu_i,\gamma_i,\theta_i)} f_i$ $(i = 1, 2)$, $\lambda_1\lambda_2 \neq 0$, $f_i \neq 0 (i = 0, 1, 2)$. Then f_{12} defined by*

$$
c_0\,\mathrm{e}^{-\frac{1}{2}D_m} f_0 \bullet f_{12} = \left[\lambda_2\,\mathrm{e}^{-\frac{1}{2}D_m} - \lambda_1\,\mathrm{e}^{\frac{1}{2}D_m}\right] f_1 \bullet f_2 \tag{4.15}
$$

with c_0 being a non-zero constant, is a new solution which is related to f_1 and f_2 under the BT (3.8)–(3.10) with parameters $(\lambda_1, \mu_1, \gamma_1, \theta_1)$, $(\lambda_2, \mu_2, \gamma_2, \theta_2)$, respectively.

Starting from the bilinear BT (4.10)–(4.12), we can derive a Lax pair for the system (4.3), (4.4). Firstly, set

$$
\psi = \frac{f}{g}, \qquad u = \ln\frac{g_{m+1}}{g}, \qquad v = \ln\frac{g_{m+1,n-1}g}{g_{m+1}g_{n-1}}
$$

in (4.10)–(4.12). Then from equations (4.10)–(4.12) we have

$$
-\mu\lambda\,\mathrm{e}^{v_{k-1}+u_{k-1}-u_{k+1}}\psi_{k+1} + 2\mu\psi_{m+1,k+1} + \alpha\psi_{m+1,k-1} - \alpha\lambda\,\mathrm{e}^{v_{k-1}}\psi_{k-1} = 0, \tag{4.16}
$$

$$
\psi_{l-1} + 4\lambda\,\mathrm{e}^{u_{l+1}-u_{m-1,l-1}}\psi_{m-1,l-1} + \theta\psi_{l+1} = 0. \tag{4.17}
$$

Through some calculations we can see that the compatibility condition of (4.16) and (4.17) generates equations (4.8) and (4.9). So they constitute a Lax pair for (4.8) and (4.9).

5. Conclusion and discussion

In this paper, we have presented two semi-discrete integrable versions and one fully discrete integrable version for the (2+1)-dimensional sinh-Gordon equation in bilinear form. Corresponding BTs, nonlinear superposition formulae and Lax pairs are derived. Based on these obtained results, it is natural to further consider other integrable properties such as determinant structures of N-soliton solutions for these semi-discrete and full-discrete integrable versions of the generalization of the sinh-Gordon equation. It is also noted that there are other recent works on the Casoratian determinant solutions to the Toda lattice and Wronskian determinant solutions to the KdV equation where through determinant formulations, some interesting solutions such as complexitons were introduced for these two equations (see, for example, [18–21]). Therefore it would be of interest to follow this line of research to derive more solutions for the bilinear (2+1)-dimensional sinh-Gordon equation and its discreted versions. Besides, it still remains unclear as to how to find similar semi-discrete and fully discrete versions for the dependent variable transformation (2.1) and (2.2) to transform the obtained semi-discrete and fully discrete bilinear equations into their nonlinear discrete versions of (1.1) and (1.2).

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Appendix A

The following bilinear identities hold for arbitrary functions a, b, c and d:

$$\left(D_y\, e^{\frac{1}{2}D_n} a \bullet a\right)\left(e^{\frac{1}{2}D_n} b \bullet b\right) - \left(D_y\, e^{\frac{1}{2}D_n} b \bullet b\right)\left(e^{\frac{1}{2}D_n} a \bullet a\right) = 2\sinh\left(\frac{D_n}{2}\right)(D_y a \bullet b) \bullet (ab) \tag{A.1}$$

$$\begin{aligned} 2D_t \cosh\left(\frac{1}{2}D_n\right)(e^{-D_m} a \bullet b) \bullet (ab) = {}& -2\sinh\left(\frac{1}{2}D_m\right)\left[\left(D_t\, e^{-\frac{1}{2}D_m+\frac{1}{2}D_n} a \bullet b\right)\right. \\ & \left.\bullet\left(e^{-\frac{1}{2}D_m-\frac{1}{2}D_n} a \bullet b\right) - \left(e^{-\frac{1}{2}D_m+\frac{1}{2}D_n} a \bullet b\right) \bullet \left(D_t\, e^{-\frac{1}{2}D_m-\frac{1}{2}D_n} a \bullet b\right)\right] \end{aligned} \tag{A.2}$$

$$\begin{aligned} &\left(D_y D_t\, e^{\frac{D_n}{2}} a \bullet a\right)\left(e^{\frac{D_n}{2}} b \bullet b\right) - \left(e^{\frac{D_n}{2}} a \bullet a\right)\left(D_y D_t\, e^{\frac{D_n}{2}} b \bullet b\right) \\ &\quad + \left(D_t\, e^{\frac{D_n}{2}} a \bullet a\right)\left(D_y\, e^{\frac{D_n}{2}} b \bullet b\right) - \left(D_t\, e^{\frac{D_n}{2}} a \bullet a\right)\left(D_y\, e^{\frac{D_n}{2}} b \bullet b\right) \\ &\quad = 2D_t \cosh\left(\frac{D_n}{2}\right)(D_y a \bullet b) \bullet (ab) \end{aligned} \tag{A.3}$$

$$\begin{aligned} &\left(D_y\, e^{\frac{D_n}{2}+D_k} a \bullet a\right)\left(e^{\frac{D_n}{2}+D_k} b \bullet b\right) - \left(D_y\, e^{\frac{D_n}{2}+D_k} b \bullet b\right)\left(e^{\frac{D_n}{2}+D_k} a \bullet a\right) \\ &\quad = 2\sinh\left(\frac{D_n}{2}+D_k\right)(D_y a \bullet b) \bullet (ab) \end{aligned} \tag{A.4}$$

$$\begin{aligned} &\left(D_t\, e^{D_m+\frac{D_n}{2}} a \bullet a\right)\left(e^{\frac{D_n}{2}} b \bullet b\right) + \left(e^{D_m+\frac{D_n}{2}} a \bullet a\right)\left(D_t\, e^{\frac{D_n}{2}} b \bullet b\right) \\ &\quad = e^{\frac{D_m}{2}}\left[\left(D_t\, e^{\frac{1}{2}D_m+\frac{1}{2}D_n} a \bullet b\right) \bullet \left(e^{-\frac{1}{2}D_m-\frac{1}{2}D_n} a \bullet b\right)\right. \\ &\qquad \left. - \left(e^{\frac{1}{2}D_m+\frac{1}{2}D_n} a \bullet b\right) \bullet \left(D_t\, e^{-\frac{1}{2}D_m-\frac{1}{2}D_n} a \bullet b\right)\right] \end{aligned} \tag{A.5}$$

$$(e^{D_1} a \bullet a)(e^{D_2} b \bullet b) - (e^{D_1} b \bullet b)(e^{D_2} a \bullet a) = 2\sinh\left(\frac{D_1 - D_2}{2}\right)\left(e^{\frac{D_1+D_2}{2}} a \bullet b\right) \bullet \left(e^{-\frac{D_1+D_2}{2}} a \bullet b\right) \tag{A.6}$$

Appendix B. Proof of propositions 2

Proof. Analogous to the deduction in [16, 17], we can show that

$$-D_y f_1\bullet f_2+(\theta_1-\theta_2)f_1f_2-2\kappa\,\mathrm{e}^{D_m}f_0\bullet f_{12}=0, \tag{B.1}$$

$$\left(\mu_1\,\mathrm{e}^{\frac{1}{2}D_m+\frac{1}{2}D_n}-\mu_2\,\mathrm{e}^{-\frac{1}{2}D_m-\frac{1}{2}D_n}\right)f_1\bullet f_2+\kappa\,\mathrm{e}^{\frac{1}{2}D_n-\frac{1}{2}D_m}f_0\bullet f_{12}=0, \tag{B.2}$$

$$\begin{aligned}0={}&-\kappa D_t\,\mathrm{e}^{-\frac{1}{2}D_m+\frac{1}{2}D_n}f_0\bullet f_{12}+\left(-\mu_1 D_t\,\mathrm{e}^{\frac{1}{2}D_m+\frac{1}{2}D_n}-\mu_2 D_t\,\mathrm{e}^{-\frac{1}{2}D_m-\frac{1}{2}D_n}\right.\\&\left.+\gamma_1\,\mathrm{e}^{\frac{1}{2}D_m+\frac{1}{2}D_n}-\gamma_2\,\mathrm{e}^{-\frac{1}{2}D_m-\frac{1}{2}D_n}\right)f_1\bullet f_2,\end{aligned} \tag{B.3}$$

$$(D_y-2\lambda_2\,\mathrm{e}^{-D_m}+\theta_2)f_1\bullet f_{12}=0, \tag{B.4}$$

$$(D_y-2\lambda_1\,\mathrm{e}^{-D_m}+\theta_1)f_2\bullet f_{12}=0. \tag{B.5}$$

$$\left(\lambda_2\,\mathrm{e}^{\frac{D_n}{2}-\frac{D_m}{2}}+\mu_2\,\mathrm{e}^{-\frac{D_n}{2}-\frac{D_m}{2}}-\mathrm{e}^{\frac{D_n}{2}+\frac{D_m}{2}}\right)f_1\bullet f_{12}=0, \tag{B.6}$$

$$\left(\lambda_1\,\mathrm{e}^{\frac{D_n}{2}-\frac{D_m}{2}}+\mu_1\,\mathrm{e}^{-\frac{D_n}{2}-\frac{D_m}{2}}-\mathrm{e}^{\frac{D_n}{2}+\frac{D_m}{2}}\right)f_2\bullet f_{12}=0. \tag{B.7}$$

Thus, in order to prove proposition 2, it suffices to show that

$$\left(D_t\,\mathrm{e}^{\frac{D_m}{2}+\frac{D_n}{2}}-\lambda_2 D_t\,\mathrm{e}^{-\frac{D_m}{2}+\frac{D_n}{2}}+\mu_2 D_t\,\mathrm{e}^{-\frac{D_m}{2}-\frac{D_n}{2}}+\mathrm{e}^{\frac{D_m}{2}+\frac{D_n}{2}}+\gamma_2\,\mathrm{e}^{-\frac{D_m}{2}-\frac{D_n}{2}}\right)f_1\bullet f_{12}=0, \tag{B.8}$$

$$\left(D_t\,\mathrm{e}^{\frac{D_m}{2}+\frac{D_n}{2}}-\lambda_1 D_t\,\mathrm{e}^{-\frac{D_m}{2}+\frac{D_n}{2}}+\mu_1 D_t\,\mathrm{e}^{-\frac{D_m}{2}-\frac{D_n}{2}}+\mathrm{e}^{\frac{D_m}{2}+\frac{D_n}{2}}+\gamma_1\,\mathrm{e}^{-\frac{D_m}{2}-\frac{D_n}{2}}\right)f_2\bullet f_{12}=0. \tag{B.9}$$

Since f_1 and f_2 are two solutions of (2.5), we have, by using (2.19), (A.3), (A.6) and $f_0\stackrel{(\lambda_i,\mu_i,\gamma_i,\theta_i)}{\longrightarrow}f_i\ (i=1,2)$, that

$$\begin{aligned}0\equiv{}&\left[\left(D_tD_y\,\mathrm{e}^{\frac{1}{2}D_n}+2D_t\,\mathrm{e}^{D_m+\frac{1}{2}D_n}-2D_t\,\mathrm{e}^{\frac{1}{2}D_n}+2\mathrm{e}^{D_m+\frac{1}{2}D_n}-2\mathrm{e}^{\frac{1}{2}D_n}\right)f_1\bullet f_1\right]\left[\mathrm{e}^{\frac{1}{2}D_n}f_2\bullet f_2\right]\\&-\left[\left(D_tD_y\,\mathrm{e}^{\frac{1}{2}D_n}+2D_t\,\mathrm{e}^{D_m+\frac{1}{2}D_n}-2D_t\,\mathrm{e}^{\frac{1}{2}D_n}+2\mathrm{e}^{D_m+\frac{1}{2}D_n}-2\mathrm{e}^{\frac{1}{2}D_n}\right)f_2\bullet f_2\right]\\&\times\left[\mathrm{e}^{\frac{1}{2}D_n}f_1\bullet f_1\right]+\left[\left(D_y\,\mathrm{e}^{\frac{1}{2}D_n}+2\mathrm{e}^{D_m+\frac{1}{2}D_n}-2\mathrm{e}^{\frac{1}{2}D_n}\right)f_1\bullet f_1\right]\left[D_t\,\mathrm{e}^{\frac{1}{2}D_n}f_2\bullet f_2\right]\\&-\left[\left(D_y\,\mathrm{e}^{\frac{1}{2}D_n}+2\mathrm{e}^{D_m+\frac{1}{2}D_n}-2\mathrm{e}^{\frac{1}{2}D_n}\right)f_2\bullet f_2\right]\left[D_t\,\mathrm{e}^{\frac{1}{2}D_n}f_1\bullet f_1\right]\\={}&2D_t\cosh\left(\frac{1}{2}D_n\right)(D_yf_1\bullet f_2)\bullet(f_1f_2)+\frac{4}{\mu_1}\sinh\left(\frac{1}{2}D_m\right)\left(\left(\mu_1\,\mathrm{e}^{\frac{1}{2}D_m+\frac{1}{2}D_n}\right.\right.\\&\left.\left.-\mu_2\,\mathrm{e}^{-\frac{1}{2}D_m-\frac{1}{2}D_n}\right)f_1\bullet f_2\right)\bullet\left(\mathrm{e}^{-\frac{1}{2}D_m-\frac{1}{2}D_n}f_1\bullet f_2\right)+\frac{4}{\mu_1}\sinh\left(\frac{1}{2}D_m\right)\\&\times\left\{\left[\left(\mu_1D_t\,\mathrm{e}^{\frac{1}{2}D_m+\frac{1}{2}D_n}+\mu_2D_t\,\mathrm{e}^{-\frac{1}{2}D_m-\frac{1}{2}D_n}\right)f_1\bullet f_2\right]\bullet\left(\mathrm{e}^{-\frac{1}{2}D_m-\frac{1}{2}D_n}f_1\bullet f_2\right)\right.\\&\left.+\left(D_t\,\mathrm{e}^{-\frac{1}{2}D_m-\frac{1}{2}D_n}f_1\bullet f_2\right)\bullet\left[\left(\mu_1\mathrm{e}^{\frac{1}{2}D_m+\frac{1}{2}D_n}-\mu_2\mathrm{e}^{-\frac{1}{2}D_m-\frac{1}{2}D_n}\right)f_1\bullet f_2\right]\right\}\\={}&-2\kappa\,\mathrm{e}^{-\frac{1}{2}D_m}\left[-\left(\mathrm{e}^{-\frac{1}{2}D_m+\frac{1}{2}D_n}f_0\bullet f_1\right)\bullet\left(D_t\,\mathrm{e}^{-\frac{1}{2}D_m-\frac{1}{2}D_n}f_2\bullet f_{12}\right)\right.\\&\left.-\left(\mathrm{e}^{-\frac{1}{2}D_m-\frac{1}{2}D_n}f_0\bullet f_1\right)\bullet\left(D_t\,\mathrm{e}^{-\frac{1}{2}D_m+\frac{1}{2}D_n}f_2\bullet f_{12}\right)\right]\\&-\frac{2\kappa\gamma_1}{\mu_1^2}\,\mathrm{e}^{-\frac{1}{2}D_m}\left[\left(\lambda_1\mathrm{e}^{-\frac{1}{2}D_m+\frac{1}{2}D_n}f_0\bullet f_1\right)\bullet\left(\mathrm{e}^{-\frac{1}{2}D_m+\frac{1}{2}D_n}f_2\bullet f_{12}\right)\right.\\&\left.-\left(\mathrm{e}^{-\frac{1}{2}D_m+\frac{1}{2}D_n}f_0\bullet f_1\right)\bullet\left(\mathrm{e}^{\frac{1}{2}D_m+\frac{1}{2}D_n}f_2\bullet f_{12}\right)\right]\\&-\frac{2\kappa}{\mu_1}\,\mathrm{e}^{-\frac{1}{2}D_m}\left[\left(\mathrm{e}^{\frac{1}{2}D_m+\frac{1}{2}D_n}f_0\bullet f_1\right)\bullet\left(D_t\,\mathrm{e}^{-\frac{1}{2}D_m+\frac{1}{2}D_n}f_2\bullet f_{12}\right)\right.\\&\left.-\left(\mathrm{e}^{-\frac{1}{2}D_m+\frac{1}{2}D_n}f_0\bullet f_1\right)\bullet\left(D_t\,\mathrm{e}^{\frac{1}{2}D_m+\frac{1}{2}D_n}f_2\bullet f_{12}\right)\right]\end{aligned}$$

$$+\frac{2\kappa}{\mu_1}\,\mathrm{e}^{-\frac{1}{2}D_m}\big(\mathrm{e}^{-\frac{1}{2}D_m+\frac{1}{2}D_n}f_0\bullet f_1\big)\bullet\big(\mathrm{e}^{\frac{1}{2}D_m+\frac{1}{2}D_n}f_2\bullet f_{12}\big)$$
$$=-\frac{2\kappa}{\mu_1}\,\mathrm{e}^{-\frac{1}{2}D_m}\big(\mathrm{e}^{-\frac{1}{2}D_m+\frac{1}{2}D_n}f_0\bullet f_1\big)\bullet\big[\big(\mu_1 D_t\,\mathrm{e}^{-\frac{1}{2}D_m-\frac{1}{2}D_n}+D_t\,\mathrm{e}^{\frac{1}{2}D_m+\frac{1}{2}D_n}$$
$$-\lambda_1 D_t\,\mathrm{e}^{-\frac{1}{2}D_m+\frac{1}{2}D_n}+\mathrm{e}^{\frac{1}{2}D_m+\frac{1}{2}D_n}+\gamma_1\,\mathrm{e}^{-\frac{1}{2}D_m-\frac{1}{2}D_n}\big)f_2\bullet f_{12}\big],\tag{B.10}$$

which implies that (B.9) holds. Similarly, we can show that (B.8) also holds. Therefore, we have completed the proof of proposition 2. □

Appendix C. Proof of proposition 4

Proof. Analogous to the deduction in [16, 17], we can show that

$$\big(\gamma_1\,\mathrm{e}^{\frac{1}{2}D_m+\frac{1}{2}D_n+D_k}-\gamma_2\,\mathrm{e}^{-\frac{1}{2}D_m-\frac{1}{2}D_n-D_k}\big)f_1\bullet f_2+k\,\mathrm{e}^{\frac{1}{2}D_n-\frac{1}{2}D_m+D_k}f_0\bullet f_{12}=0,\tag{C.1}$$
$$D_y f_1\bullet f_2=-2k\,\mathrm{e}^{D_m}f_0\bullet f_{12}+(\theta_1-\theta_2)f_1 f_2,\tag{C.2}$$
$$(D_y-2\lambda_2\,\mathrm{e}^{-D_m}+\theta_2)f_1\bullet f_{12}=0,\tag{C.3}$$
$$(D_y-2\lambda_1\mathrm{e}^{-D_m}+\theta_2)f_2\bullet f_{12}=0,\tag{C.4}$$
$$\big(-\lambda_1\,\mathrm{e}^{-\frac{D_m}{2}+\frac{D_n}{2}+D_k}+2\mathrm{e}^{\frac{D_m}{2}+\frac{D_n}{2}+D_k}+\gamma_1\,\mathrm{e}^{-\frac{D_m}{2}-\frac{D_n}{2}-D_k}\big)f_0\bullet f_1=0,\tag{C.5}$$
$$\big(-\lambda_2\,\mathrm{e}^{-\frac{D_m}{2}+\frac{D_n}{2}+D_k}+2\mathrm{e}^{\frac{D_m}{2}+\frac{D_n}{2}+D_k}+\gamma_2\,\mathrm{e}^{-\frac{D_m}{2}-\frac{D_n}{2}-D_k}\big)f_0\bullet f_2=0.\tag{C.6}$$

Thus, in order to prove proposition 4, it suffices to show that

$$\big(\gamma_2\,\mathrm{e}^{-\frac{1}{2}D_m-\frac{1}{2}D_n-D_k}+2\mathrm{e}^{\frac{1}{2}D_m+\frac{1}{2}D_n+D_k}-\lambda_2\,\mathrm{e}^{-\frac{1}{2}D_m+\frac{1}{2}D_n+D_k}\big)f_1\bullet f_{12}=0,\tag{C.7}$$
$$\big(\gamma_1\,\mathrm{e}^{-\frac{1}{2}D_m-\frac{1}{2}D_n-D_k}+2\mathrm{e}^{\frac{1}{2}D_m+\frac{1}{2}D_n+D_k}-\lambda_1\,\mathrm{e}^{-\frac{1}{2}D_m+\frac{1}{2}D_n+D_k}\big)f_2\bullet f_{12}=0.\tag{C.8}$$

Since f_1 and f_2 are two solutions of (3.4), we have, by using (C.2), (C.6), (A.1), (A.6) and $f_0\overset{(\lambda_i,\mu_i,\gamma_i,\theta_i)}{\longrightarrow}f_i$ $(i=1,2)$, that

$$0\equiv\big[\big(D_y\,\mathrm{e}^{\frac{1}{2}D_n+D_k}+4\mathrm{e}^{D_m+\frac{1}{2}D_n+D_k}-4\mathrm{e}^{\frac{1}{2}D_n+D_k}\big)f_1\bullet f_1\big]\big(\mathrm{e}^{\frac{1}{2}D_n+\frac{1}{2}D_k}f_2\bullet f_2\big)$$
$$-\big[\big(D_y\,\mathrm{e}^{\frac{1}{2}D_n+D_k}+4\mathrm{e}^{D_m+\frac{1}{2}D_n+D_k}-4\mathrm{e}^{\frac{1}{2}D_n+D_k}\big)f_2\bullet f_2\big]\big(\mathrm{e}^{\frac{1}{2}D_n+\frac{1}{2}D_k}f_1\bullet f_1\big)$$
$$=2\sinh\left(\frac{1}{2}D_n+D_k\right)(D_y f_1\bullet f_2)\bullet(f_1 f_2)$$
$$+8\sinh\left(\frac{1}{2}D_m\right)\big(\mathrm{e}^{\frac{1}{2}D_m+\frac{1}{2}D_n+D_k}f_1\bullet f_2\big)\bullet\big(\mathrm{e}^{-\frac{1}{2}D_m-\frac{1}{2}D_n-D_k}f_1\bullet f_2\big)$$
$$=4k\sinh\left(\frac{1}{2}D_n+D_k\right)(\mathrm{e}^{-D_m}f_0\bullet f_{12})\bullet(f_1 f_2)-\frac{8}{\gamma_2}\sinh\left(\frac{1}{2}D_m\right)$$
$$\times\big(\mathrm{e}^{\frac{1}{2}D_m+\frac{1}{2}D_n+D_k}f_1\bullet f_2\big)\bullet\big[\big(\gamma_1\,\mathrm{e}^{\frac{1}{2}D_m+\frac{1}{2}D_n+D_k}-\gamma_2\,\mathrm{e}^{-\frac{1}{2}D_m-\frac{1}{2}D_n-D_k}\big)f_1\bullet f_2\big]$$
$$=4k\sinh\left(\frac{1}{2}D_n+D_k\right)(\mathrm{e}^{-D_m}f_0\bullet f_{12})\bullet(f_1 f_2)$$
$$+\frac{8k}{\gamma_2}\sinh\left(\frac{D_m}{2}\right)\big(\mathrm{e}^{\frac{1}{2}D_m+\frac{1}{2}D_n+D_k}f_1\bullet f_2\big)\bullet\big(\mathrm{e}^{\frac{1}{2}D_n-\frac{1}{2}D_m+D_k}f_0\bullet f_{12}\big)$$
$$=2k\,\mathrm{e}^{\frac{1}{2}D_m}\big[\big(\mathrm{e}^{-\frac{1}{2}D_n-\frac{1}{2}D_m-D_k}+\frac{2}{\gamma_2}\,\mathrm{e}^{\frac{1}{2}D_m+\frac{1}{2}D_n+D_k}-\frac{\lambda_2}{\gamma_2}\,\mathrm{e}^{-\frac{1}{2}D_m+\frac{1}{2}D_n+D_k}\big)f_1\bullet f_2\big]$$
$$\bullet\big(\mathrm{e}^{-\frac{1}{2}D_m+\frac{1}{2}D_n+D_k}f_0\bullet f_2\big),$$

which implies that (C.7) holds. Similarly, we can show that (C.8) also holds. Therefore, we have completed the proof of proposition 4. □

Appendix D. Proof of propositions 6

Proof. Analogous to the deduction in [16, 17], we can show that

$$(\theta_1\,\mathrm{e}^{-D_l}-\theta_2\,\mathrm{e}^{D_l})f_1\bullet f_2=4c_0\,\mathrm{e}^{-D_m+D_l}f_0\bullet f_{12}, \tag{D.1}$$

$$\left(\frac{\theta_1}{\lambda_1}\mathrm{e}^{-D_l-\frac{1}{2}D_m}-\frac{\theta_2}{\lambda_2}\mathrm{e}^{D_l+\frac{1}{2}D_m}\right)f_1\bullet f_2+\frac{c_0}{\lambda_1\lambda_2}\mathrm{e}^{-D_l-\frac{1}{2}D_m}f_0\bullet f_{12}=0, \tag{D.2}$$

$$\left(\frac{\mu_2}{\lambda_2}\mathrm{e}^{-\frac{1}{2}D_n}-\frac{\mu_1}{\lambda_1}\mathrm{e}^{\frac{1}{2}D_n}\right)f_1\bullet f_2+\frac{c_0}{\lambda_1\lambda_2}\mathrm{e}^{\frac{1}{2}D_n}f_0\bullet f_{12}=0, \tag{D.3}$$

$$\left(\mu_2\,\mathrm{e}^{-\frac{1}{2}D_m-\frac{1}{2}D_n}-\mu_1\,\mathrm{e}^{\frac{1}{2}D_m+\frac{1}{2}D_n}\right)f_1\bullet f_2+c_0\,\mathrm{e}^{-\frac{1}{2}D_m+\frac{1}{2}D_n}f_0\bullet f_{12}=0. \tag{D.4}$$

Thus, in order to prove proposition 6, it suffices to show that

$$\left(\lambda_1\,\mathrm{e}^{\frac{D_n}{2}-\frac{D_m}{2}}+\mu_1\,\mathrm{e}^{-\frac{D_n}{2}-\frac{D_m}{2}}-\mathrm{e}^{\frac{D_n}{2}+\frac{D_m}{2}}\right)f_2\bullet f_{12}=0, \tag{D.5}$$

$$\left(\lambda_2\,\mathrm{e}^{\frac{D_n}{2}-\frac{D_m}{2}}+\mu_2\,\mathrm{e}^{-\frac{D_n}{2}-\frac{D_m}{2}}-\mathrm{e}^{\frac{D_n}{2}+\frac{D_m}{2}}\right)f_1\bullet f_{12}=0, \tag{D.6}$$

$$(\mathrm{e}^{-D_l}+4\lambda_1\,\mathrm{e}^{-D_m-D_l}+\theta_1\,\mathrm{e}^{D_l})f_2\bullet f_{12}=0, \tag{D.7}$$

$$(\mathrm{e}^{-D_l}+4\lambda_2\,\mathrm{e}^{-D_m-D_l}+\theta_2\,\mathrm{e}^{D_l})f_1\bullet f_{12}=0, \tag{D.8}$$

$$\left(-\lambda_1\,\mathrm{e}^{-\frac{D_m}{2}+\frac{D_n}{2}+D_k}+2\mathrm{e}^{\frac{D_m}{2}+\frac{D_n}{2}+D_k}+\alpha_1\,\mathrm{e}^{\frac{-D_m}{2}-\frac{D_n}{2}-D_k}\right)f_2\bullet f_{12}=0, \tag{D.9}$$

$$\left(-\lambda_2\,\mathrm{e}^{-\frac{D_m}{2}+\frac{D_n}{2}+D_k}+2\mathrm{e}^{\frac{D_m}{2}+\frac{D_n}{2}+D_k}+\alpha_2\,\mathrm{e}^{\frac{-D_m}{2}-\frac{D_n}{2}-D_k}\right)f_1\bullet f_{12}=0. \tag{D.10}$$

By use of (D.1) and (D.2), we have

$$\begin{aligned}
0&=\left[\left(\mathrm{e}^{\frac{1}{2}D_n+D_l}-5\mathrm{e}^{\frac{1}{2}D_n-D_l}+4\mathrm{e}^{D_m+\frac{1}{2}D_n+D_l}\right)f_1\bullet f_1\right]\left(\mathrm{e}^{-\frac{1}{2}D_n+D_l}f_2\bullet f_2\right)\\
&\quad-\left[\left(\mathrm{e}^{\frac{1}{2}D_n+D_l}-5\mathrm{e}^{\frac{1}{2}D_n-D_l}+4\mathrm{e}^{D_m+\frac{1}{2}D_n+D_l}\right)f_2\bullet f_2\right]\left(\mathrm{e}^{-\frac{1}{2}D_n+D_l}f_1\bullet f_1\right)\\
&=2\sinh\left(\frac{1}{2}D_n\right)(\mathrm{e}^{D_l}f_1\bullet f_2)\bullet(\mathrm{e}^{-D_l}f_1\bullet f_2)\\
&\quad+8\sinh\left(\frac{1}{2}D_m+\frac{1}{2}D_n\right)(\mathrm{e}^{\frac{1}{2}D_m+D_l}f_1\bullet f_2)\bullet(\mathrm{e}^{-\frac{1}{2}D_m-D_l}f_1\bullet f_2)\\
&=\frac{2}{\theta_1}\sinh\left(\frac{1}{2}D_n\right)(\mathrm{e}^{D_l}f_1\bullet f_2)\bullet(4c_0\,\mathrm{e}^{-D_m-D_l}f_0\bullet f_{12})\\
&\quad-\frac{8\lambda_1}{\theta_1}\sinh\left(\frac{1}{2}D_m+\frac{1}{2}D_n\right)(\mathrm{e}^{\frac{1}{2}D_m+D_l}f_1\bullet f_2)\bullet\left(\frac{-c_0}{\lambda_1\lambda_2}\mathrm{e}^{-D_l-\frac{1}{2}D_m}f_0\bullet f_{12}\right)\\
&=\frac{4c_0}{\theta_1}\mathrm{e}^{D_l+\frac{1}{2}D_m}\left[\left(\mathrm{e}^{\frac{1}{2}D_n-\frac{1}{2}D_m}+\frac{\mu_2}{\lambda_2}\mathrm{e}^{-\frac{1}{2}D_m-\frac{1}{2}D_n}-\frac{1}{\lambda_2}\mathrm{e}^{\frac{1}{2}D_m+\frac{1}{2}D_n}\right)f_1\bullet f_{12}\right]\\
&\quad\bullet\left(\mathrm{e}^{-\frac{1}{2}D_m-\frac{1}{2}D_n}f_0\bullet f_2\right),
\end{aligned} \tag{D.11}$$

so we have proved that (D.5) holds, similarly we can show that (D.6) holds. By use of (D.3) and (D.4), we can show that (D.7) and (D.8) hold. By use of (D.1) and (D.2), from equation (4.4) we can show that (D.9) and (D.10) hold. So we have completed proposition 6. □

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